Generalized Family of Multi-Step Utility Functions for Adoption in UNDP's Human Development index

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Abstract

UNDP introduced the human development report in 1991, a radical change in its approach in terms of multi-step fragmenting of the utility function and a concept of threshold income level, which was followed till 1998 by following an ill-structured Atkinson based multi-step utility function. Realizing the weaknesses in it formulation, the UNDP abandoned the approach abruptly from 1999 onwards. As a viable substitute for erroneous Atkinson based multi-step formulation of utility function used in UNDP's human development reports till 1998, we provided a generalized family of the utility functions under the premise of multi-step formulation while adhering to the concept of threshold income level and have showed in the present study that the earlier proposed two alternative formulations due to Bhatnagar (2001, 2002a) turn out to be particular cases.

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Development is a complex and multi-dimensional process. Growth of per capita gross national product (GNP) initially emerged as the commonly used index of development with a view to take into consideration the ability of any country to expand its output more briskly vis-à-vis the growth rate of its population. Subsequently, it was realized that though a large number of third world countries recorded satisfactory growth rate in per capita GNP, yet their masses failed to evince a matching amelioration during 1950s and 1960s in as far as their actual standard of leading a healthy life fraught with educational attainments was concerned. Evidently, the experience thus gathered signaled that there existed some inherent problems with the way the development was being defined and actually, there prevailed no automatic link between human development parse and economic growth.

Realization about the limitations of per capita real GDP as a measure of welfare became instrumental to the commencement of efforts towards developing measures based on social indicators. United Nations Research Institute for Social Development (UNRISD) (1970) in its study initially began with 73 social and economic variables with a view to finally select the most appropriate indicators of socioeconomic development by following the analysis of relationships between these indicators at different levels of development. A synthetic index of development was constructed as a more representative and sensitive than the per capita GNP. The UNRISD study first reduced the list of socioeconomic variables from 73 to 42 at the first instance and then to 16

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highly correlated core variables at the end. The core indicators included nine social indicators and seven economic indicators. Approaches like physical quality of life indicators (PQLI) and basic needs indicators were introduced in 1970s and 1980s, for further details on which one may refer to Streeten (1977), Morris (1979), Hicks and Streeten (1979), and Larson and Wilford (1979).

The decade of 1990s gave birth to the concept of the human development index (HDI), which was presented by United Nations Development Programme (UNDP) in its human development reports (HDR) for the first time in 1990. The HDI proposed by UNDP in HDR 1990, according to Trabold-Nübler (1991), was an attempt to place the emphasis on human welfare rather than on progress of the national economy. The human development reports (HDRs) are being brought out by UNDP on an annual basis ever since 1990, which besides visualizing human development as the process of increasing people's options, have also been ranking different countries according to their values of HDI.

The human development reports (HDRs) brought out by UNDP have promulgated a new vision on measuring development by combining indicators (a) depicting people's desire for 'leading a long and healthy life' as measured by life expectancy at birth, (b) 'to be knowledgeable' as measured by a synthetic combination of adult literacy rate and the combined gross enrolment ratio for primary, secondary, and tertiary schools, and (c) 'ensuring decent standard of living' as measured by per capita GDP (in PPP$) into a composite human development index (HDI), which would serve as a frame of reference for both social and economic development of any country.

Research Implications and Brief Account of the Methodological Refinements During the First Decade of Inception

HDI has played two key roles in the field of applied development economics viz., (a) as a tool to popularize human development as a new understanding of well-being, and (b) as an alternative to GDP per capita as a way to measure levels of development for comparison across both countries and time. The HDI has generated a lot of curiosity and interest among policy makers, development professionals, academicians, press, and the public alike. There existed certain inherent conceptual weaknesses in the manner in which Atkinson's concave well-being function (of the form [$y^{(1-\varepsilon)}/(1-\varepsilon)$, $0\varepsilon\ \varepsilon<1$]) was handled by UNDP in formulation a multi-step utility function ever since 1991 till 1998. Oblivion of those conceptual issues hovering around faulty formulations of UNDP's Atkinson based multi-step utility function, the experts of UNDP methodology could not visualize resolution thereof while remaining within the premises of the multi-step utility function and threshold income level. Various modifications brought about by the UNDP for the computation of the HDI in the HDRs since inception have analytically been studied in literature (Hopkins , 1991 ; McGillivray, 1991 ; McGillivray & White, 1993 ; Noorbakhsh, 1998) with a focus on the methodological aspects regarding construction of synthetic measure of educational attainment and the treatment given to formulation of utility function under income component of the HDI for ensuring diminishing marginal returns from the real per capita GDP levels.

Methodology for computation of HDI has seen tremendous changes, especially during the decade of the nineties, since being in the initial stages of evolution, the methodology suffered from a variety of conceptual problems during that period, an insight into the genesis of the need for modifications in the formulation of the HDI and particularly its utility function has been provided by Bhatnagar (2001) and Bhatnagar (2002a). The conceptual issues pertaining to the problems of non-homogeneity of 'units & dimensions' of various terms in the Atkinson based multi-step utility function and the non-fulfillment of the principle of diminishing marginal utility have been highlighted by him with illustrations and further cited by Majumder and Kusago (2012) while coming up with their different alternative formulation, though outside the ambit of multi-step formulation of utility function.

The approach followed by Bhatnagar (2001) and Bhatnagar (2002a) has been multi-faceted namely, to (a) identify different conceptual flaws in the UNDP's methodology of computing the HDI during the nineties, (b) provide remedial measures for their resolution, (c) attempt to link those remedies with a proper rationale for modifications, which were actually carried out by the UNDP in the HDRs from time to time under a different pretext during the period of the nineties, (d) develop other viable alternatives free from bugs, and (e) study the variations in the ranks of the countries under the various options of interest to researchers and academicians.

An overview of the UNDP's methodology presented by Bhatnagar (2001) categorizes entire decade of the nineties into (a) nascent phase in the evolutionary process during the initial period from 1990 to 1993, (b) the year 1994 as the turning point in the methodology by virtue of introducing fixed normative values for maximum and minimum goal posts for various indicators, (c) the period from 1995 to 1998 as the initial phase of the stabilized methodology, and (d) circumlocutory approach from 1999 onwards. The component of educational attainment was not properly handled in the UNDP's HDR (1991). While trying to merge two quantities, one of which was expressed in percent terms and another was a pure number, the HDR (1991) did not realize the conceptual problem that the additive operation would not be tenable with terms of dissimilar units of measurement. Accordingly, the computation of index for educational attainment in the HDR (1991) was conceptually incorrect. However, subsequently, in the HDR (1992), for obtaining the educational attainment index, the two factors of adult literacy and mean years of schooling were first converted into unit-free quantities before merging with each other. Although modification was carried out by the UNDP in the HDR (1992), but the underlying need for bringing about the change in the UNDP's methodology remained missing in the technical literature of UNDP. The analytical review by Bhatnagar (2001) has inter-alia aided in capturing the conceptual rationale behind the abrupt switching over from improper combination of adult literacy percentage with mean years of schooling while working out the indicator for educational attainment in HDR (1992).

In HDR (1990), the UNDP used a truncated logarithmic utility function initially, giving zero weights to the income levels above a particular cut-off level. The diminishing marginal utility was yielded by the truncated logarithmic function up to the cut-off level only, and any additional income level beyond the cut-off level yielded no additional utility. The implication of the formulation of the utility function in the HDR (1990) lacked the logical instinct and the conceptual appeal. During 1991 to 1998, the UNDP in its HDRs utilized a multi-step utility function with Atkinson's formulation as radix for capturing the extent of utility derived from any levels of real per capita GDP of countries. The effect of diminishing marginal utility was built into the formulation beyond a certain level of real per capita GDP income termed as threshold income level such that as long as the real per capita GDP level of income for any country remained below the threshold income level, no 'discounting' or 'adjustment' was necessitated. The range beyond the threshold income level was sub-divided by the UNDP into income segments of equal width; the span of each income segment being equal to the numerical value of the threshold income level. The Atkinson based multi-step utility function was defined differently for various income segments beyond the threshold income level, and the 'adjusted' value of utility was computed using the appropriate formula corresponding to the particular income segment. While ensuring conformity to the principle of the diminishing marginal utility, the above practice and procedure was prescribed by the UNDP for all its HDRs from 1991 to 1998 for working out the 'discounted' or 'adjusted' value of utility from the Atkinson based multi-step utility function. However, in all its subsequent HDRs from 1999 onwards, UNDP discarded the Atkinson based multi-step utility function and switched over to the non-truncated logarithmic function of real per capita GDP income levels for formulation of the utility function.

Bhatnagar's Alternative Formulations of Multi-Step Utility Functions

The UNDP's methodology in respect of component of income has undergone drastic changes in terms of formulations of utility function used for 'adjusting' values of real per capita GDP income, the details of which have been compacted by Bhatnagar (2002b) (on page 254) and further have been cited by Liptak (2009). The availability of Choubey's (1998) two-step formulation in the literature initially emerged as a ray of hope for exploring the possibility of evolving improved methodology even in the case of multi-step formulation of the utility function. Consequently, Bhatnagar (2001) considered the multi-step decomposition of the full range of income values beyond the threshold income level, by taking the immediately next income segment beyond the threshold income as of width numerically equal to the value of the threshold income, while each subsequent income segment thereafter was of equal width equal to twice the numerical value of the threshold income level, and proposed a viable alternative multi-step formulation termed as Bhatnagar's first alternative formulation (abbreviated as, BFAF) in place of the erroneous Atkinson based multi-step utility function, without deviating from the framework of the threshold income level and multi-step formulation of the utility function. The precise formulation of BFAF while considering the fragmentation of the entire range of real per capita GDP for different countries in any HDR into income-bands as 0 to y^*, y^* to $2y^*$, $2y^*$ to $4y^*$, $4y^*$ to $6y^*$, $6y^*$ to $8y^*$ and so on the basis of threshold income level (say, y^*) has been proposed in literature [See equations (16A) to (16F) in Bhatnagar (2001), p.56] and can be re-written as under :

(1.1A) $W_0 = y,$ for $0 \varepsilon y \varepsilon y^*$

(1.1B) $W_1 = y^* + y^* \log(y/y^*),$ for $y^* < y \varepsilon 2y^*$

(1.1C) $W_2 = y^* + (1/2)y^*(\log 2) + (1/2)y^* \log(y/y^*),$ for $2y^* < y \varepsilon 4y^*$

(1.1D) $W_4 = y^* + (1/4)y^*(2\log 2 + \log 4) + (1/4)y^* \log(y/y^*),$ for $4y^* < y \varepsilon 6y^*$

(1.1E) $W_6 = y^* + (1/8)y^*(4\log 2 + 2\log 4 + \log 6) + (1/8)y^* \log(y/y^*),$ for $6y^* < y \varepsilon 8y^*$

(1.1F) $W_{2n} = (\varepsilon^{n}_{m=0}(1/2^m)\, y^* \log(2m)) + (1/2^n)y^* \log(y/y^*),$ for $2ny^* < y \;\; 2(n+1)y^*$ and $n \varepsilon 1$

Choubey's (1998) alternative in a simplified situation of two-step formulation of the utility function became a particular case of BFAF. The BFAF has been found to be free from the problem of mathematical tenability of the additive operations of any non-homogenous terms involved in the formulation of the utility function and comprehensively adheres to the principle of the diminishing marginal returns. Kelley's (1991) observations about taking the HDI as a linear function of a logarithm of real per capita GDP has been found to be holding true under BFAF.

It may be observed that the first two income bands namely 0 to y^* and y^* to $2y^*$ are of same width y^*, but the subsequent income bands though being of the same width, have double the span of the first two intervals. However, by keeping the width of all income bands formed on the basis of the threshold income as same, Bhatnagar's second alternative formulation (abbreviated as BSAF) appears in the literature (see equations (7A) to (7H) in Bhatnagar (2002a), pp.135-136) as another viable substitute for the Atkinson-based multi-step utility function. Following the analogous notations as earlier, Bhatnagar's second alternative formulation can be re-written as:

(1.2A) $U_0 = y,$ for $0 \varepsilon y \varepsilon y^*$

(1.2B) $U_1 = y^* + (1/2)y^* \log(y/y^*),$ for $y^* < y \varepsilon 2y^*$

(1.2C) $U_2 = y^* + (1/6)y^*(\log 2) + (1/3)y^* \log(y/y^*),$ for $2y^* < y \varepsilon 3y^*$

(1.2D) $U_3 = y^* + (1/6)y^*(\log 2) + (1/12)y^*(\log 3) + (1/4)y^* \log(y/y^*),$ for $3y^* < y \varepsilon 4y^*$

(1.2E) $U_4 = y^* + (1/6)y^*(\log 2) + (1/12)y^*(\log 3) + (1/20)y^*(\log 4) + (1/5)y^* \log(y/y^*),$ for $4y^* < y \varepsilon 5y^*$

(1.2F) $U_5 = y^* + (1/6)y^*(\log 2) + (1/12)y^*(\log 3) + (1/20)y^*(\log 4) + (1/30)y^*(\log 5) + (1/6)y^* \log(y/y^*),$ for $5y^* < y \varepsilon 6y^*$

(1.2G) $U_{n-1} = y^* + (1/6)y^*(\log 2) + (1/12)y^*(\log 3) + (1/20)y^*(\log 4) + (1/30)y^*(\log 5) +$
$\ldots + (1/(n-1)n)y^*(\log(n-1)) + (1/n)y^* \log(y/y^*),$ for $(n-1)y^* < y \varepsilon ny^*$

(1.2H) $U_n = y^* + (1/6)y^*(\log 2) + (1/12)y^*(\log 3) + (1/20)y^*(\log 4) + (1/30)y^*(\log 5) +$
$\ldots + (1/(n(n+1))y^*(\log n) + (1/(n+1))y^* \log(y/y^*),$ for $ny^* < y \varepsilon (n+1)y^*$

The above general equation (1.2H) can compactly be written as :

$$(1.2\mathrm{I}) \quad U_n = y^* + (\varepsilon^{\,n}_{m=1} \{1/(m\,(m+1)\}y^*\log(m)) + \{1/(n+1)\}y^*\log(y/y^*), \qquad \text{for } ny^* < y \ \varepsilon \ (n+1)y^*$$

The BSAF proposed by Bhatnagar (2002a) has been shown by him to be free from the two shortcomings regarding mathematical formulation and the principle of the diminishing marginal utility.

Current Status on BFAF and BSAF Approaches

Bhatnagar and Tiwari (2014) studied the comparisons within each of the four HDRs during 1995 to 1998 between the ranks offered by BFAF and BSAF on the one hand and post-1998s transformed approach (being termed here as post-Atkinson based approach) of UNDP on the other hand for South Asian countries in the Indian sub-continent; the variations in the ranks of some of the industrially advanced countries of the World were also presented. Using Atkinson's well known class of concave functions widely used in economic literature on inequality as increasing function with limiting value as zero, the concept of Kakwani's achievement indicator was introduced by Bhatnagar and Tiwari (2009) to define Kakwani's achievement indicators for life expectancy at birth, adult literacy rate, combined gross enrolment ratios and real per capita GDP to compute the modified values of HDI with a view to study the sensitization in the ranking of various countries. Following the similar weighting method as adopted by the UNDP in respect of its educational attainment index, the adult literacy rate and combined gross enrolment ratios were combined together for each country with two-third and one-third weights under Kakwani's approach. While impacts on the ranks of various countries covered in the human developments reports (HDRs) due to BFAF have been studied by Bhatnagar (2002b) for the period from 1995 to 1998, during which the UNDP's methodology in constructing the human development index (HDI) had remained more or less stable, the rank-sensitization of countries in the HDRs during the same period under Kakwani (1993) achievement indicator approach was also examined by Bhatnagar and Tiwari (2009).

To ascertain the impact on the ranks of the countries, when under Atkinson based multi-step utility function, varying threshold income levels of other HDR (like HDR 1997, HDR 1996, and HDR 1995) were applied to the same fixed dataset of real per capita GDP pertaining to HDR 1998 for 'discounting' the income levels higher than the chosen threshold income level. The variations in the ranks of the countries during 1995 to 1998 on account of the two modifications namely, BFAF and BSAF in the utility function were comprehensively explored by Bhatnagar and Tiwari (2014) under different scenarios.

Bhatnagar and Tiwari (2014), while appraising the two formulations namely, BFAF and BSAF concluded that since the width of income segments formed of the entire range, real per capita GDP happens to be smaller for BSAF as compared to that for BFAF, the numbers of countries experiencing gain in their ranks or suffering from a decline in their ranks with reference to UNDP's original ranks of the HDRs also turn out to be correspondingly smaller for the BFAF as compared to the BSAF. Furthermore, the countries gaining ranks due to BSAF are common with those gaining ranks due to BFAF, while the gaining uncommon countries between the two approaches are those countries which are exclusively affected with upward movement in the ranks by the BFAF only. Similarly, the countries losing ranks due to BSAF are common with those losing ranks due to BFAF, while the losing uncommon countries between the two approaches are those countries which are exclusively affected with downward movement in the ranks by the BFAF only. It has also been shown that all the countries in any HDR, which are affected on account of upward/ downward movement of their ranks under BSAF are also affected, experiencing similar upward/ downward movement of ranks (though not by the same magnitude of variation) when BFAF is applied to the same HDR data. Having studied the individual potential of BFAF and BSAF vis-à-vis the Atkinson based multi-step utility function adopted by the UNDP in the HDRs from 1995 to 1998, Bhatnagar and Tiwari (2014) compared and contrasted the two formulations amongst themselves, by first computing the fresh HDI-ranks for all the countries while considering BSAF at the first instance and thereafter changing the formulation to BFAF to work out the revised ranks for them. Though both the formulations yield different rankings for most of the countries in the HDR, but the magnitude of variations on either side remains modest and does not become quite high. The variations in ranks of the prominently affected countries in the HDR

(1998), the HDR (1997), the HDR (1996), and the HDR (1995), on changing BSAF with BFAF while retaining the threshold income level intact were also studied by Bhatnagar and Tiwari (2014). The effect on the ranks of the countries due to replacement of the UNDP's Atkinson based multi-step utility function with BFAF and BSAF while using the same HDR dataset but varying the levels of threshold income was further studied.

In this paper, we concentrate more closely on Bhatnagar's first and second alternative formulations of utility function, that is BFAF and BSAF, with a view to present (in the subsequent sections), the generalized family of multi-step utility functions.

Generalization of Bhatnagar's First Alternative Formulation as (B, T)- Family (Type-I) of Multi-Step Utility Functions

Taking *'B'* and *'T'* as two parameters, we introduce (B,T)- family (Type-I) of multi-step formulations of utility function as under:

(1.3A) $U_0 = y,$ $\quad 0 y \, \varepsilon \, y^*$

(1.3B) $U_1 = y^* + y^* \log(y/y^*),$ $\quad y^* < y \, \varepsilon \, 2y^*$

(1.3C) $U_B = K_{2B-T} + \{1/(T-B)^{B/2}\} y^* \log[B(T-B)/2] + \{1/(T-B)^{B/2}\} y^* \log(y/y^*),$ $\quad$ for $By^* < y \, \varepsilon \; Ty^*$ & $B = 2$;

such that K_α is the value of function U_α at point $y = y^*$.

It may be noted that equations (1.1A) and (1.1B) are identical to equations (1.3A) and (1.3B) respectively. Equation (1.3C) designates the family of multi-step formulations of utility function with two parameters as *'B'* and *'T'*. It would now suffice to show that for specific values of *'B'* and *'T'*, the equation (1.1C) to (1.1F) turns out to be particular cases of equation (1.3C).

Let us take values of *'B'* as even numbers from 2 onwards. Further, the values of *'T'* should be chosen as *'B'* augmented by 2. Accordingly, we can write:

(1.4A) $B = 2, 4, 6, 8, \ldots$ so on.

(1.4B) $T = B + 2$

We now begin by first taking *'B'* as 2 with corresponding value of *'T'* as 4. Evidently, the income interval $By^* < y \; \varepsilon \; Ty^*$ then refers to $2y^* < y \; \varepsilon \; 4y^*$ and the equation (1.3C) reduces as under:

(1.5A) $U_2 = K_0 + (1/2) y^* \log 2 + (1/2) y^* \log(y/y^*)$

Using equation (1.3A) and taking K_0 as the value of U_0 at $y = y^*$, we obtain $K_0 = y^*$, which on substituting in equation (1.5A) yields the following equation for $2y^* < y \; \varepsilon \; 4y^*$:

(1.5B) $U_2 = y^* + (1/2) y^* \log 2 + (1/2) y^* \log(y/y^*)$

Note that the equations (1.5B) and (1.1C) are identical and thus (B, T) -family (Type-I) reduces to equation (1.1C) as a particular case for the income interval $2y^* < y \; \varepsilon \; 4y^*$. Using equation (1.5B), the value of K_2 is thus obtained as:

$K_2 = y^* + (1/2) y^* \log 2 + (1/2) y^* \log(y^*/y^*)$

Or that

(1.5C) $K_2 = y^* + (1/2) y^* \log 2$

On considering *'B'* as 4 with resulting value of *'T'* as 6, for corresponding income interval $4y^* < y \varepsilon 6y^*$, the Equation (1.3C) accordingly reduces to:

(1.5D) $U_4 = K_2 + (1/4)y^* \log 4 + (1/4)y^* \log(y/y^*)$

Substituting from equation (1.5C) in (1.5D) and re-arranging the terms, we get for income interval $4y^* < y \varepsilon 6y^*$

(1.5E) $U_4 = y^* + (1/4)y^*(2\log 2 + \log 4) + (1/4)y^* \log(y/y^*)$

Note that the equations (1.5E) and (1.1D) are identical and thus (B, T) - family (Type-I) includes equation (1.1D) as a particular case for the income interval $4y^* < y \varepsilon 6y^*$. Using equation (1.5E) and substituting $y = y^*$, K_4 is obtained as:

$K_2 = y^* + (1/4)y^*(2\log 2 + \log 4) + (1/4)y^* \log 1$

Or that

(1.5F) $K_2 = y^* + (1/4)y^*(2\log 2 + \log 4)$

Again with value of *'B'* taken as 6 and corresponding value of *'T'* as 8, for income interval $6y^* < y \varepsilon 8y^*$, we get from the equation (1.3C) :

(1.5G) $U_6 = K_4 + (1/8)y^* \log 6 + (1/8)y^* \log(y/y^*)$

Substituting from equation (1.5F) for the income interval namely, $6y^* < y \varepsilon 8y^*$, the equation (1.5G) simplifies as:

(1.5H) $U_6 = y^* + (1/8)y^*(4\log 2 + 2\log 4 + \log 6) + (1/8)y^* \log(y/y^*)$

Since the equations (1.5H) and (1.1E) are identical, (B,T)-family (Type-I) reduces to equation (1.1E) as a particular case for the income interval $6y^* < y \varepsilon 8y^*$. Proceeding as before, the equation (1.5H) provides the value of K_6 as:

(1.5I) $K_6 = y^* + (1/8)y^*(4\log 2 + 2\log 4 + \log 6)$

For income interval $8y^* < y \varepsilon 10y^*$, we now need to take *'B'* and *'T'* as 8 and 10 respectively in equation (1.3C) to obtain:

(1.5J) $U_8 = K_6 + (1/16)y^* \log 8 + (1/16)y^* \log(y/y^*)$

Using equation (1.5I) and (1.5J), on re-arranging the terms we get, for the income interval $8y^* < y \varepsilon 10y^*$

(1.5K) $U_8 = y^* + (1/8)y^*(4\log 2 + 2\log 4 + \log 6) + (1/16)y^* \log 8 + (1/16)y^* \log(y/y^*)$

Or that :

$U_8 = y^* + (1/2)y^*(\log 2) + (1/4)y^*(\log 4) + (1/8)y^*(\log 6) + (1/16)y^* \log 8 + (1/16)y^* \log(y/y^*)$

Or that :

$U_8 = (1/2^0)y^* \log(0) + (1/2^1)y^* \log(2x1) + (1/2^2)y^* \log(2x2) + (1/2^3)y^* \log(2x3) + (1/2^4)y^* \log(2x4) + (1/2^4)y^* \log(y/y^*)$

Or that :

(1.5L) $U_8 = (\varepsilon^4_{m=0}(1/2^m)y^* \log(2m)) + ((1/2^4)y^* \log(y/y^*)$

If we consider *'B'* as $2n$ and *'T'* as $2n + 2$, similar to equation (1.5L), we shall get the final equation for income interval of $2ny^* < y \; (2n + 2)y^*$, as :

(1.5M) $U_{2n} = (\varepsilon^{n}_{m=0}(1/2^m)\, y^* \log(2m)) + ((1/2^n)y^* \log(y/y^*)$

Evidently, the equations (1.5M) and (1.1F) are identical, and thus (*B, T*)- family (Type-I) reduces to equation (1.1F) as a particular case for any income interval $2ny^* < y \; \varepsilon \; (2n + 2)\, y^*$. Having observed the phenomenon in general for all $B = 2$, thus (*B,T*) - Family (Type-I) of multi-step formulation contains Bhatnagar's first alternative formulation as a particular case.

Generalization of Bhatnagar's Second Alternative Formulation as (B,T)-Family (Type-II) of Multi-Step Utility Functions

We introduce (*B,T*)- family (Type-II) of multi-step formulations of utility function as under :

(1.6A) $U_0 = y \quad , 0 \; \varepsilon \; y \; \varepsilon \; y^*$

(1.6B) $U_B = K_{2B-T} + \{(T - B)/BT\}y^* \log \mathrm{B} + \{1/(T)^{(T-B)}\}y^* \log(y/y^*)$, for $By^* < y \; \varepsilon \; Ty^*$ & $B = 1$;
such that K_α is the value of function U_α at point $y = y^*$; *'B'* & *'T'* are two parameters.

Clearly, equations (1.2A) and (1.6A) are identical. It would now suffice to show that for specific values of *'B'* and *'T'*, the equations (1.2B) to (1.2I) are obtained as the particular cases of equation (1.6B). Let us take values of *'B'* as natural integers from 1 onwards and the values of *'T'* chosen as *'B'* incremented by 1. Thus, we can write :

(1.7A) $B = 1, 2, 3, 4, 5$ ……so on.
(1.7B) $T = B + 1$

We note that the value of K_0 from equation (1.6A) is obtained as y^* by putting $y = y^*$. First, taking *'B'* as 1 with corresponding value of *'T'* as 2, the equation (1.6B) for $1y^* < y \; \varepsilon \; 2y^*$ provides:

(1.8A) $U_1 = K_0 + \{(2\text{-}1)/1\mathrm{x}2\}y^* \log 1 + \{1/2^{(2-1)}\}y^* \log(y/y^*)$

Or that :

$U_1 = K_0 + (1/2)\, y^* \log(y/y^*)$

Or that :

(1.8B) $U_1 = y^* + (1/2)\, y^* \log(y/y^*)$

Note that the equations (1.8B) and (1.2B) are identical, and thus, (*B,T*)- family (Type-II) reduces to equation (1.2B) as a particular case for the income interval $y^* < y \; \varepsilon \; 2y^*$. Further, using equation (1.8B), the value of K_1 is obtained as :

$K_1 = y^* + (1/2)y^* \log(y^*/y^*)$

Or that :

(1.8C) $K_1 = y^*$

Let us now consider *'B'* as 2 with corresponding value of *'T'* as 3. Evidently, the income interval $By^* < y \; \varepsilon \; Ty^*$ then refers to $2y^* < y \; \varepsilon \; 3y^*$, and the equation (1.6B) simplifies to:

(1.8D) $U_2 = K_1 + \{(3\text{-}2)/2x3\}y^* \log 2 + \{1/3^{(3-2)}\}y^* \log(y/y^*)$

Or that :

$U_2 = K_1 + (1/6)y^* \log 2 + (1/3)y^* \log(y/y^*)$

Or that :

(1.8E) $U_2 = y^* + (1/6)y^* \log 2 + (1/3)y^* \log(y/y^*)$

Evidently, equations (1.8E) and (1.2C) turn out to be identical, implying that (B,T)- family (Type-II) reduces to equation (1.2B) as a particular case for the income interval $2y^* < y \varepsilon\ 3y^*$. Equation (1.8E) provides the value of K_2 by substituting $y = y^*$.

(1.8F) $K_2 = y^* + (1/6)y^* \log 2$

Taking *'B'* as 3 with corresponding value of *'T'* as 4, for the income interval $3y^* < y \varepsilon\ 4y^*$, the equation (1.6B) becomes:

$U_3 = K_2 + \{(4\text{-}3)/3x4\}y^* \log 3 + \{1/4^{(4-3)}\}y^* \log(y/y^*)$

Or that :

$U_3 = K_2 + (1/12)y^* \log 3 + (1/4)y^* \log(y/y^*)$

Or that :

(1.8G) $U_3 = y^* + (1/6)y^* \log 2 + (1/12)y^* \log 3 + (1/4)y^* \log(y/y^*)$

Evidently, equations (1.8G) and (1.2D) being identical, it emerges out that (B,T) -family (Type-II) reduces to equation (1.2D) as a particular case for the income interval $3y^* < y \varepsilon\ 4y^*$. In the similar way, the value of K_3 is obtained as:

(1.8H) $K_3 = y^* + (1/6)y^* \log 2 + (1/12)y^* \log 3$

Using the value of K_3 from the equation (1.8H) in equation (1.6B) after taking *'B'* and *'T'* as 4 and 5 respectively, we obtain for $4y^* < y \varepsilon\ 5y^*$

(1.8I) $U_4 = y^* + (1/6)y^* \log 2 + (1/12)y^* \log 3 + (1/20)y^* \log 4 + (1/5)y^* \log(y/y^*)$

Since equations (1.8I) and (1.2E) are identical, clearly (B, T)-family (Type-II) reduces to equation (1.2E) as a particular case for the income interval $4y^* < y \varepsilon\ 5y^*$. We next compute the value of K_4 from equation (1.8I) by putting $y = y^*$.

(1.8J) $K_4 = y^* + (1/6)y^* \log 2 + (1/12)y^* \log 3 + (1/20)y^* \log 4$

In equation (1.6B), let us now put $B = 5$ and $T = 6$ to obtain :

$U_5 = K_4 + \{(5\text{-}4)/4x5\}y^* \log 4 + \{1/5^{(5-4)}\}y^* \log(y/y^*)$

On substituting value of K_4, the above equation simplifies for income interval $5y^* < y\ \varepsilon\ 6y^*$ as under :

(1.8K) $U_5 = y^* + (1/6)y^* \log 2 + (1/12)y^* \log 3 + (1/20)y^* \log 4 + (1/30)y^* \log 5 + (1/6)y^* \log(y/y^*)$

Since the equations (1.8K) and (1.2F) are identical, thus (B,T)-family (Type-II) reduces to equation (1.2F) as a particular case for the income interval $5y^* < y\ \varepsilon\ 6y^*$. Equation (1.8K) can be further compacted to be re-written as under :

$U_5 = y^* + \{1/(1x2)\}y^* \log(1) + \{1/(2x3)\}y^* \log(2) + \{1/(3x4)\}y^* \log(3) + \{1/(4x5)\}y^* \log(4) + \{1/(5x6)\}y^* \log(5) + \{1/(5+1)\}y^* \log(y/y^*)$

Or that :

(1.8L) $U_5 = y^* + (\varepsilon^{5}_{m=1}\{1/(m(m+1)\}y^* \log(m)) + \{1/(5+1)\}y^* \log(y/y^*)$

If we consider *'B'* as *n* and *'T'* as (*n* +1), similar to equation (1.8L), we shall get the final equation for income interval of $ny^* < y \ (n+1)y^*$, as:

(1.8M) $U_n = y^* + (\varepsilon^{n}_{m=1}\{1/(m(m+1)\}y^* \log(m)) + \{1/(n+1)\}y^* \log(y/y^*)$

Evidently, the equations (1.5M) and (1.2I) are identical, implying that (*B,T*)-family (Type-II) reduces to equation (1.2I) as a particular case for any income interval $ny^* < y \ (n+1)y^*$. Having observed the phenomenon in general for all $B = 1$, thus (*B,T*) - Family (Type-II) of multi-step formulation includes Bhatnagar's second alternative formulation as a particular case.

Generalized Family of Multi-Step Utility Functions Including both (B,T)-Families (Type-I & II) as Special Cases

Consider a generalized (*B,T*)- Family of multi-step formulations of utility function, being introduced now as under :

(1.9A)	$U_0 = y,$	for $0 \ \varepsilon \ y \ \varepsilon \ y^*$
(1.9B)	$U_B = K_{2B-T} + \lambda y^* \log \mu + (1/v) y^* \log(y/y^*),$	for $By^* < y \ \varepsilon \ Ty^*$; & $B = 1$;

such that K_α is the value of function U_α at point $y = y^*$; & λ, μ, ν, *'B'* & *'T'* are parameters.

We now show below that (*B,T*) - Families (Type-I & Type-II) delineated in the earlier sections are particular cases of the generalized (*B,T*)- Family of multi-step formulation of the utility functions.

(*B,T*) - Family (Type-I) of Multi-step Utility Function as a Particular Case

For $B \geq 2$, when we take $\lambda = \{1/(T-B)^{B/2}\}$, $\mu = B(T-B)/2$, and $\nu = (T-B)^{B/2}$, equation (1.9B) reduces to :

(1.9C) $U_B = K_{2B-T} + \{1/(T-B)^{B/2}\}y^* \log[B(T-B)/2] + \{1/(T-B)^{B/2}\}y^* \log(y/y^*),$ for $By^* < y \ Ty^*$ & $B = 2$.

Equations (1.9C) and (1.3C) are identical, which implies that (*B,T*) - family (Type-I) of multi-step utility function occurs as a particular category of generalized (*B,T*)- Family, when *B* =2. When we take $\lambda = (T-B)/BT$, $\mu = B$, and $\nu = (T/2)$, the equation (1.9B) reduces to :

$U_B = K_{2B-T} + \{(T-B)/BT\}y^* \log B + (2/T)y^* \log(y/y^*)$

The particular choices of *'B'* as 1 and *'T'* as *B* +1 refers to income interval of $1y^* < y \ \varepsilon \ 2y^*$ for which we obtain from above a simplified equation as :

$U_1 = K_0 + \{(2-1)/(1x2)\}y^* \log 1 + (2/2)y^* \log(y/y^*)$

Or that :

(1.9D) $U_1 = K_0 + y^* \log(y/y^*)$

The value of K_0 is obtained by putting $y = y^*$ in equation (1.9A). Accordingly, we have $K_0 = y^*$. Substituting the value of K_0 in equation (1.9D), we finally obtain for income interval $1y^* < y \ \varepsilon\ 2y^*$

(1.9E) $U_1 = y^* + y^* \log (y/y^*)$

The equation (1.9E) is identical to (1.3C) for the income band $1y^* < y\ \varepsilon\ 2y^*$, which implies (B,T)- family (Type-I) of multi-step utility function occurs as a particular category of generalized (B,T) - Family, when $B = 1$ and $T = B + 1$. Of course, for the income interval $0\ \varepsilon\ y\ \varepsilon\ y^*$, the two equations namely (1.9A) and (1.3A) are already identical. Thus, the generalized (B,T)- Family of multi-step formulation is still more general and contains *(B,T)* - family (Type-I) as a particular case.

(*B,T*)- Family (Type-II) of Multi- Step Utility Function- as a Particular Case

For $B \geq 1$, when we take $\lambda = \{(T-B)/BT\}$, $\mu = B$ and $\nu = (T)^{(T-B)}$, equation (1.9B) reduces to :

(1.9F) $U_B = K_{2B-T} + \{(T-B)/BT\} y^* \log B + \{1/(T)^{(T-B)}\} y^* \log(y/y^*)$, for $By^* < y\ \varepsilon\ Ty^*$ & $B = 1$

Equations (1.9F) and (1.6B) are identical, which implies that (B,T)- family (Type-II) of multi-step utility function occurs as a particular category of generalized (B,T) - Family, when $B = 1$. The equation (1.9A) is obviously identical to (1.6A), which implies (B,T) - family (Type-II) of multi-step utility function occurs as a particular category of generalized (B,T)- Family for the income band $0\ \varepsilon\ y\ \varepsilon\ 1y^*$.

Having observed both the cases, that is, (a) $B = 1$ and (b) $B = 1$, thus generalized (B,T) - Family of multi-step formulation is still more general and contains (B,T)- family (Type-II) also as a particular case.

Conclusion

The methodology for constructing the utility function for the income component in the human development index initially began with a truncated logarithmic function of per capita GDP income in UNDP's human development report of 1990, but for the purpose of 'adjusting' the actual values for different countries, the adoption of Atkinson-based multi-step formulation of income-transforming-utility function coupled with the concept of threshold income level, which remained in vogue during 1991 to 1998, emerged out to be a landmark change before finally again reverting from 1999 onwards to logarithmic utility function, but as a non-truncated one as against the truncated logarithmic utility function.

It should be seen with a sense of great astonishment that the concept of multi-step formulation of utility function, which was so close to the heart of UNDP was abruptly deserted by the UNDP from 1999 itself without trying out any other viable substitute within the premise of multi-step formulation. Since alternative formulations like Bhatnagar's first and the second alternative formulations (BFAF and BSAF) now already exist in literature as viable replacements of UNDP's faulty Atkinson-based multi-step utility function without abandoning the premise of multi-step formulation, and further without dropping the concept of threshold income level, one could logically conclude that UNDP should continue with the multi-step utility function by deploying formulations proposed by Bhatnagar (2001 & 2002a) instead of the presently followed un-truncated logarithmic utility function. This paper introduces a still more generalized family of multi-step utility functions, which connotes both BFAF and BSAF as its particular cases.

Limitations of the Study and Scope for Further Research

Realizing the need for UNDP to switch over again to the premise of multi-step formulation of the utility function from its present approach, more studies on variations and sensitization of ranks of different countries would be required. Amongst all plausible values for the parameter 'ε' lying between zero and unity, Bhatnagar and Tiwari (2009) considered three sets of illustrative values for 'ε' as 1/2, 1/3, and 2/3 for the computation purposes since their objective involved only the assessment of relative ranking of different countries and taking any different value for the parameter 'ε' would not be of material significance. However, certain other phenomena can be studied with other different values of parameter 'ε'.

One can further study variations in ranks of the countries under various other scenarios namely, (a) when under Kakwani's achievement approach, Choubey (1998)'s two-step utility function is replaced by either Bhatnagar's first or second alternative formulation; (b) when Choubey's (1998) two-step utility function without Kakwani's achievement approach is replaced by Choubey's two-step utility function under Kakwani's achievement approach; (c) when Choubey's (1998) two-step utility function without Kakwani's achievement approach is replaced either by Bhatnagar' first or second alternative formulation; (d) when any one of Choubey's (1998), Bhatnagar's first or second alternative formulation is replaced by UNDP's traditional multi-step formulation within the framework of Kakwani's achievement approach; (e) when Bhatnagar's first alternative formulation is by Bhatnagar's second alternative formulation within framework of Kakwani's achievement approach; and (e) when Bhatnagar's second alternative formulation is by Bhatnagar's first alternative formulation within the framework of Kakwani's achievement approach.

References

Bhatnagar, R. K. (2001). An analysis of the evolution of the human development index with special reference to its income component. *The Bangladesh Development Studies, 27*(3), 35-65.

Bhatnagar, R. K. (2002a). Problems with UNDP multi-step utility function and their resolutions: Generalization of some results. *The Indian Journal of Economics, 83*(328), 123-138.

Bhatnagar, R. K. (2002b). Constructing the Human Development Index: Effect of the adaptive multi-step formulation of utility function : A research note. *South Asian Economic Journal, 3* (2), 253-263. DOI: 10.1177/139156140200300209

Bhatnagar, R. K., & Tiwari, N. (2009). Rank sensitization of countries in human development reports of UNDP in view of Kakwani's approach for achievement indicator. *Journal of Reliability and Statistical Studies, 2* (1), 71-84.

Bhatnagar, R. K., & Tiwari, N. (2014). Impact of Bhatnagar's first & second alternative formulations on rankings of the countries in UNDP's human development reports during 1995 to 1998. *International Journal of Education, Development, Society and Technology (IJEDST), 2* (1), 1-18.

Choubey, P.K. (1998). Income utility curve: Flaws in UNDP formulation. *Indian Journal of Economics, 78* (311), 515-523.

Hicks, N., & Streeten, P. (1979). Indicators of development: The search for a basic needs yardstick. *World Development, 7*(6), 567-580.

Hopkins, M. (1991). Human development revisited: A UNDP report. *World Development, 19* (10), 1469-1473.

Kakwani, N. (1993). Performance in living standards: An international comparison. *Journal of Development Economics, 41* (2), 307-336.

Kelley, A.C. (1991). The human development index: “Handle with care.” *Population and Development Review, 17*(2), 315-324.

Larson, D.A., & Wilford, W.T. (1979). The physical quality of life index: A useful social indicator? *World Development, 7*(6), 581-584. DOI: 10.1016/0305-750X(79)90094-9

Lipták, K. (2009). *Development or decline? Determination of human development at subregional level with the estimation of HDI*. EU Working Papers, 4/2009, Retrieved from http://epa.oszk.hu/00000/00026/00045/pdf/euwp_EPA00026_2009_04_087-103.pdf

Majumder, A., & Kusago, T. (2012). A note on methodology of treating income in human development index. *The Indian Economic Journal, 60* (2), 57-72.

McGillivray, M. (1991). The human development index: Yet another redundant composite development indicator? *World Development, 19* (10), 1461-1468.

McGillivray, M., & White, H. (1993). Measuring development? The UNDP's human development index. *Journal of International Economics, 5*(2), 183-192.

Morris D., Morris (1979). *Measuring the condition of world's poor. The physical quality of life index.* Frank Cass, London.

Noorbakhsh, F. (1998). The human development index: Some technical issues and alternative indices. *Journal of International Development, 10* (5), 589-605.

Streeten, P. (1977). The distinctive features of a basic needs approach to development. *International Development Review, 19* (3), 8-16.

Trabold-Nübler, H. (1991). The human development indexA new development indicator? *Intereconomics, 26* (5), 236 - 243. DOI : 10.1007/BF02928996

ZUNRISD. (1970). *UNRISD contents and measurement of socio-economic development* (p. 63). Geneva.